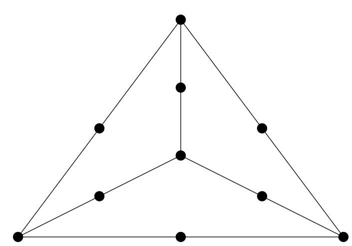
Resit Graph Theory

9 April 2025, 15.00–17.00

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Always make sure to state clearly any results from the lecture notes you are using.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (20 pts).
Determine a maximum matching in the following graph, and use the Berge–Tutte formula to prove that it is a maximum matching. (As mentioned above, make sure to clearly state any results from the lecture notes you are using.)



Exercise 2 (20 pts)
Let G be a non-empty planar graph that does not contain any three-cycles. Show that $2e(G) \leq 4v(G) - 8$.
Bonus (10 pts): Use this to show that $\chi(G) \leq 4$.

Exercise 3 (30 pts).
(a) Show that for k odd, any k-regular graph has an even number of vertices.
(b) Let G be a non-empty 3-regular Hamiltonian graph. Show that $\chi'(G) = 3$.

Exercise 4 (20 pts)
Let $1 \leq k < n$ be integers and let $f(n,k)$ be the minimum number such that every graph G on n vertices with at least $f(n,k)$ edges is k-connected. Show $f(n,k) = \binom{n-1}{2} + k$.
Hint: To show that $f(n,k) \leq \binom{n-1}{2} + k$, consider a graph with at least $\binom{n-1}{2} + k$ edges. How many edges are missing? Use this to show that there are no vertices $u, v, w_1, \ldots, w_{k-1}$ such that u is not connected to v in $G \setminus \{w_1, \ldots, w_{k-1}\}$.

(The end)